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Branching processes in the presence of random immigration and representations for time series

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Abstract. The statistics of branching processes of particles in a multiplicative medium are developed by taking account of a random non-stationary particle immigration. The probability distribution functions of the number of particles found in the medium at any fixed time and/or of particles counted in a time interval are given in closed form. Also given are conditional probabilities, conditioned on having a fixed number of particles at time $t = 0$. These are applied to the case that exactly two particles are produced by branching.

1. Introduction

Branching processes or multiplicative processes have been studied in many fields for a long time. The early stochastic studies were done for the processes starting from a single particle or ancestor (Kendall 1948, Bellman and Harris 1952). The branching processes with immigration of particles from outside sources were first investigated by Sevastyanov (1957) and time-varying immigration was introduced by Hering (1973). In recent work, emigration as well as immigration has been taken into consideration (Pakes 1986).

The application of branching processes was done to investigate the fluctuation of neutron number in a nuclear reactor in which neutron chain reactions take place (Bell 1965, Kobayashi 1968) and the fluctuation of electron number by electron multiplicative processes in solids (Lin 1979, 1981). In the field of high-energy physics, the observed particle spectra have also been studied by using statistics of Markov branching processes (Giovannini 1979, Carruthers and Shin 1985, Shin 1986). In these studies, however, the equations for the probability generating function (PGF) are not solved in closed form except for a very few cases, and the probability distribution functions are in general obtained only for the case that the branching processes are started with some particles injected at the initial time. This holds also for several recent works on fractals (Lushinikov *et al* 1981) and $1/f$ noise (Furukawa 1986).

As can be seen from these works, the statistics of branching processes have been investigated enthusiastically, but only limited effort has been put into the investigation of the effect of random immigration and into that of time series using the branching model (Heyde 1986). In the present work, therefore, the statistics of branching processes in the presence of random immigration are developed and the representations for time series are derived for a particular case that reaction rates of particles in the medium are constant.

2. Branching started from a single particle

Consider a medium in which a particle may be subjected to capture and multiplicative reactions. By a multiplicative reaction, several particles appear in the medium and a branch is produced. We suppose in the present work that the system is homogeneous, i.e. the system is well stirred or the velocity of particles is infinitely large.

2.1. Existing particles

Suppose no particle existed in the medium before the time $t=0$, and one particle has been injected at $t=0$. We consider then the probability $p(n, t)$ that n particles are found in the medium at time $t>0$. This had been already considered by Kobayashi (1968) in the argument for the fluctuation of neutron number in a nuclear reactor, which is described briefly in the following. The equation for $p(n, t)$ is easily obtained as

$$\frac{\mathrm{d}p(n,t)}{\mathrm{d}t}=\lambda_{\mathrm{c}}(n+1)p(n+1,t)-\lambda_{\mathrm{t}}np(n,t)+\lambda_{\mathrm{m}}\sum_{\nu=2}^{n}p_{\nu}(n-\nu+1)p(n-\nu+1,t) \tag{1}$$

where λ_{c} and λ_{m} are probabilities of capture and multiplicative reactions, respectively, in unit time for a particle,

$$\lambda_{\mathrm{t}}=\lambda_{\mathrm{c}}+\lambda_{\mathrm{m}} \tag{2}$$

and p_{ν} is the distribution function of the number of particles produced by a multiplicative reaction. It is normalised to $\Sigma_{\nu=2}^{\infty}p_{\nu}=1$. Introducing the PGF

$$h(w,t)=\sum_{n=0}^{\infty}p(n,t)w^{n} \tag{3}$$

and

$$\psi(w)=\sum_{\nu=2}^{\infty}p_{\nu}w^{\nu} \tag{4}$$

equation (1) can be expressed as

$$\frac{\partial h(w,t)}{\partial t}=(\lambda_{\mathrm{m}}\psi(w)-\lambda_{\mathrm{t}}w+\lambda_{\mathrm{c}})\frac{\partial h(w,t)}{\partial w}=\Phi(w)\frac{\partial h(w,t)}{\partial w} \tag{5}$$

where

$$\Phi(w)=\lambda_{\mathrm{m}}\psi(w)-\lambda_{\mathrm{t}}w+\lambda_{\mathrm{c}}. \tag{6}$$

The initial and boundary conditions for $h(w, t)$ are

$$h(w,0)=w \tag{7}$$

and

$$h(1,t)=1 \tag{8}$$

respectively. When λ_{c} and λ_{m} are constant, equation (5) is solved in closed form by the Laplace transformation so that

$$h(w,t)=F[t+k(w)] \tag{9}$$

where

$$k(w) = \int \frac{\mathrm{d}w}{\Phi(w)} \tag{10}$$

and $F(w)$ is the inverse function of $k(w)$, i.e. $F[k(w)] = w$ due to (7). Equation (8) is satisfied since $k(1) = \infty$ and $k(\infty) = 1$.

2.2. Counted particles

In order to consider the statistics of particles counted by a detector in the medium, we introduce the probability $p_m(n, t)$ that the detector counts m particles during a time interval $(0, t)$ and n particles are found in the medium at time $t > 0$ when one particle has been injected at $t = 0$. The PGF of $p_m(n, t)$ is given by

$$g(v, w, t) = \sum_{n,m=0}^{\infty} \sum_{n=0}^{\infty} p_m(n, t) v^m w^n. \tag{11}$$

Assuming the detector to be of absorption type, equations equivalent to (1) and (5) are, respectively,

$$\frac{\mathrm{d}p_m(n, t)}{\mathrm{d}t} = \lambda_a (n+1) p_m(n+1, t) - \lambda_t n p_m(n, t)$$
$$+ \lambda_m \sum_{\nu=2}^{n} p_\nu (n - \nu + 1) p_m(n - \nu + 1, t) + \lambda_d (1 - \delta_{m,0})(n+1) p_{m-1}(n+1, t) \tag{12}$$

and

$$\frac{\partial g(v, w, t)}{\partial t} = (\lambda_m \psi(w) - \lambda_t w + \lambda_a + \lambda_d v) \frac{\partial g}{\partial w} = \phi(v, w) \frac{\partial g}{\partial w} \tag{13}$$

where λ_d is the detection rate for a particle, λ_a describes capture of particles in the medium other than absorption by the detector,

$$\lambda_t = \lambda_c + \lambda_m = \lambda_a + \lambda_d + \lambda_m \tag{14}$$

and

$$\phi(v, w) = \lambda_m \psi(w) - \lambda_t w + \lambda_a + \lambda_d v. \tag{15}$$

The PGF $g(v, w, t)$ satisfies the conditions

$$g(v, w, 0) = w \tag{16}$$

$$g(1, 1, t) = 1 \tag{17}$$

and

$$g(1, w, t) = h(w, t). \tag{18}$$

The solution of (13) is

$$g(v, w, t) = I(v, t + K(v, w)) \tag{19}$$

where

$$K(v, w) = \int \frac{\mathrm{d}w}{\phi(v, w)} \tag{20}$$

and $I(v, w)$ is the inverse function of $K(v, w)$ with regard to w, i.e. $I[v, K(v, w)] = w$.

The PGF of the probability $p_m(t)$ that m particles are counted during the time interval $(0, t)$ is given by

$$f(v, t) = \sum_{m=0}^{\infty} p_m(t) v^m = g(v, 1, t). \tag{21}$$

3. Branching with random immigration

When a random source exists in the medium, all the individual particles from the source can produce branches and, therefore, many branches having different origins are found in the medium.

3.1. Existing particles

The PGF of the probability $P(n, t)$ that n particles are found in the medium at any time

$$H(w, t) = \sum_{n=0}^{\infty} P(n, t) w^n \tag{22}$$

is obtained by multiplying all the PGF for the individual source particles (Bell 1965), which is described by

$$H(w, t) = \exp\left(\int_{t_0}^{t} S(\tau)(h(w, t-\tau) - 1)\, \mathrm{d}\tau \right) \tag{23}$$

where $S(\tau)$ is the random immigration rate of source particles. Notice that the source is assumed to be present since $t = t_0$, and the initial state at $t = t_0$ is assumed to be empty.

3.2. Counted particles

Consider the probability $P_m(t, \Delta t)$ that m particles are counted during a time interval $(t, t+\Delta t)$, and define its PGF as

$$L(v, t, \Delta t) = \sum_{m=0}^{\infty} P_m(t, \Delta t) v^m. \tag{24}$$

The probability that no source particle appears in the medium during the time interval (t_0, t) is

$$P_s = \exp\left(-\int_{t_0}^{t} S(\tau)\, \mathrm{d}\tau \right). \tag{25}$$

Consequently, $P_s S(\tau_1)\, \mathrm{d}\tau_1$, $P_s S(\tau_1)\, \mathrm{d}\tau_1 S(\tau_2)\, \mathrm{d}\tau_2, \ldots$, are, respectively, the probabilities that one, two, ..., particles appear in the medium in infinitesimal time intervals around the times τ_1, τ_1 and $\tau_2, \ldots$. Supposing that a source particle which appears at τ_i

($\tau_i < t$) results in n_i particles at time t and they contribute to the count in $(t, t+\Delta t)$, the PGF of count probabilities due to source particles appearing in (t_0, t) is, therefore,

$$P_s \sum_{l=0}^{\infty} \int_{t_0}^{t} S(\tau_1) \sum_{n_1=0}^{\infty} p(n_1, t-\tau_1)(f(v, \Delta t))^{n_1}\, d\tau_1 \ldots$$

$$\times \int_{T_{l-1}}^{t} S(\tau_l) \sum_{n_l=0}^{\infty} p(n_l, t-\tau_l)(f(v, \Delta t))^{n_l}\, d\tau_l$$

$$= P_s \sum_{l=0}^{\infty} \int_{t_0}^{t} S(\tau_1) h(f(v, \Delta t), t-\tau_1)\, d\tau_1 \ldots \int_{T_{l-1}}^{t} S(\tau_l) h(f(v, \Delta t), t-\tau_l)\, d\tau_l. \tag{26}$$

In a similar way, the PGF of count probabilities due to source particles appearing in $(t, t+\Delta t)$ is described as

$$\exp\left(-\int_{t}^{t+\Delta t} S(\tau)\, d\tau\right) \sum_{l=0}^{\infty} \int_{t}^{t+\Delta t} S(\tau_1) f(v, t+\Delta t-\tau_1)\, d\tau_1 \ldots$$

$$\times \int_{T_{l-1}}^{t+\Delta t} S(\tau_l) f(v, t+\Delta t-\tau_l)\, d\tau_l. \tag{27}$$

Since

$$\int_{t_1}^{t_2} y(\tau_1)\, d\tau_1 \int_{\tau_1}^{t_2} y(\tau_2)\, d\tau_2 \ldots \int_{\tau_{l-1}}^{t_2} y(\tau_l)\, d\tau_l = \frac{1}{l!}\left(\int_{t_1}^{t_2} y(\tau)\, d\tau\right)^l \tag{28}$$

for a function $y(t)$, (26) and (27) become

$$P_s\left[\sum_{l=0}^{\infty} \frac{1}{l!}\left(\int_{t_0}^{t} S(\tau) h(f(v, \Delta t), t-\tau)\, d\tau\right)^l\right]$$

$$= \exp\left(\int_{t_0}^{t} S(\tau)(h(f(v, \Delta t), t-\tau)-1)\, d\tau\right)$$

$$= H(f(v, \Delta t), t) \tag{29}$$

and

$$\exp\left(-\int_{t}^{t+\Delta t} S(\tau)\, d\tau\right)\left[\sum_{l=0}^{\infty} \frac{1}{l!}\left(\int_{t}^{t+\Delta t} S(\tau) f(v, t+\Delta t-\tau)\, d\tau\right)^l\right]$$

$$= \exp\left(\int_{t}^{t+\Delta t} S(\tau)(f(v, t+\Delta t-\tau)-1)\, d\tau\right) \tag{30}$$

respectively. The PGF $L(v, t, \Delta t)$ is obtained by multiplying (29) and (30):

$$L(v, t, \Delta t) = H(f(v, \Delta t), t) \exp\left(\int_{t}^{t+\Delta t} S(\tau)(f(v, t+\Delta t-\tau)-1)\, d\tau\right). \tag{31}$$

4. Time series representations

We consider, in this section, conditional probabilities which are applicable to describe time series.

4.1. *Existing particles*

Suppose $P(m; n, t)$ is the probability that n particles are found in the medium at time $t>0$ after we had m particles at $t=0$. From a similar consideration around (25), the probability $P(m; n, t)$ is given by

$$P(m; n, t) = \exp\left(-\int_0^t S(\tau)\,\mathrm{d}\tau\right)\left(\Sigma^{(0)} \prod_{i=1}^{m} p(n_i, t) + \Sigma^{(1)} \prod_{i=1}^{m} p(n_i, t) \int_0^t S(\tau) p(k, t-\tau)\,\mathrm{d}\tau + \Sigma^{(2)} \prod_{i=1}^{m} p(n_i, t) \int_0^t S(\tau_1) p(k_1, t-\tau_1)\,\mathrm{d}\tau_1 \times \int_{\tau_1}^t S(\tau_2) p(k_2, t-\tau_2)\,\mathrm{d}\tau_2 + \ldots\right) \tag{32}$$

where $\Sigma^{(i)}$ is a sum over $n_1, n_2, \ldots, n_m$, $k_1, k_2, \ldots$, and k_i constrained to

$$n_1 + n_2 + \ldots + n_m + k_1 + k_2 + \ldots + k_i = n.$$

If we multiply (32) by u^m and w^n and sum up over m and n, its $(l+1)$th term is written by

$$\begin{aligned}
&\exp\left(-\int_0^t S(\tau)\,\mathrm{d}\tau\right) \sum_{m=0}^{\infty} u^m \sum_{n=0}^{\infty} w^n \Sigma^{(l)} \prod_{i=1}^{m} p(n_i, t) \\
&\quad\times \int_0^t S(\tau_1) p(k_1, t-\tau_1)\,\mathrm{d}\tau_1 \ldots \int_{\tau_{l-1}}^t S(\tau_l) p(k_l, t-\tau_l)\,\mathrm{d}\tau_l \\
&= \exp\left(-\int_0^t S(\tau)\,\mathrm{d}\tau\right) \sum_{m=0}^{\infty} u^m \sum_{n=0}^{\infty} \Sigma^{(l)} \prod_{i=1}^{m} p(n_i, t) w^{n_i} \\
&\quad\times \int_0^t S(\tau_1) p(k_1, t-\tau_1) w^{k_1}\,\mathrm{d}\tau_1 \ldots \int_{\tau_{l-1}}^t S(\tau_l) p(k_l, t-\tau_l) w^{k_l}\,\mathrm{d}\tau_l \\
&= \exp\left(-\int_0^t S(\tau)\,\mathrm{d}\tau\right) \sum_{m=0}^{\infty} u^m \prod_{i=1}^{m} \left(\sum_{n_i=0}^{\infty} p(n_i, t) w^{n_i}\right) \\
&\quad\times \int_0^t S(\tau_1) \sum_{k_1=0}^{\infty} p(k_1, t-\tau_1) w^{k_1}\,\mathrm{d}\tau_1 \ldots \int_{\tau_{l-1}}^t S(\tau_l) \sum_{k_l=0}^{\infty} p(k_l, t-\tau_l) w^{k_l}\,\mathrm{d}\tau_l \\
&= \exp\left(-\int_0^t S(\tau)\,\mathrm{d}\tau\right) \sum_{m=0}^{\infty} [u h(w, t)]^m \\
&\quad\times \int_0^t S(\tau_1) h(w, t-\tau_1)\,\mathrm{d}\tau_1 \ldots \int_{\tau_{l-1}}^t S(\tau_l) h(w, t-\tau_l)\,\mathrm{d}\tau_l.
\end{aligned} \tag{33}$$

Introducing the PGF

$$G(u, w, t) = \sum_{m=0}^{\infty} u^m \sum_{n=0}^{\infty} w^n P(m; n, t) \tag{34}$$

and using (28) and (33),

$$G(u, w, t) = \exp\left(-\int_0^t S(\tau)\,\mathrm{d}\tau\right)\left(\sum_{m=0}^{\infty}(uh(w, t))^m\right)\left[\sum_{l=0}^{\infty}\frac{1}{l!}\left(\int_0^t S(\tau)h(w, t-\tau)\,\mathrm{d}\tau\right)^l\right]$$

$$= \sum_{m=0}^{\infty}(uh(w, t))^m \exp\left(\int_0^t S(\tau)(h(w, t-\tau)-1)\,\mathrm{d}\tau\right). \tag{35}$$

For the PGF defined by (34), we can restrict to

$$0 \leqslant u \leqslant 1 \qquad \text{and} \qquad 0 \leqslant w \leqslant 1 \tag{36}$$

and, therefore,

$$0 \leqslant uh(w, t) < 1 \tag{37}$$

unless both u and w become unity at the same time. In this case, (35) is expressed as

$$G(u, w, t) = \frac{1}{1-uh(w, t)}\exp\left(\int_0^t S(\tau)(h(w, t-\tau)-1)\,\mathrm{d}\tau\right)$$

$$= \frac{1}{1-uh(w, t)}Q(w, t) \tag{38}$$

where

$$Q(w, t) = \exp\left(\int_0^t S(\tau)(h(w, t-\tau)-1)\,\mathrm{d}\tau\right). \tag{39}$$

As can be seen from the above considerations, the coefficient $Q(w, t)$ gives the effect of source particles appearing in $(0, t)$ and the other part of $G(u, w, t)$ is that of particles existing at $t = 0$.

The conditional probability is given by

$$P(m; n, t) = \frac{1}{m!\,n!}\left(\frac{\partial^m}{\partial u^m}\frac{\partial^n}{\partial w^n}G(u, w, t)\right)_{u=w=0}$$

$$= \frac{1}{n!}\left(\frac{\partial^n}{\partial w^n}(h(w, t))^m Q(w, t)\right)_{w=0}$$

$$= \frac{1}{n!}\left(\frac{\partial^n}{\partial w^n}G_m(w, t)Q(w, t)\right)_{w=0}$$

$$= \sum_{k=0}^{n}\frac{1}{(n-k)!\,k!}(G_m^{(n-k)})_{w=0}(Q^{(k)})_{w=0} \tag{40}$$

where

$$G_m(w, t) = (h(w, t))^m. \tag{41}$$

Using Leibniz's formula, the $(n-k)$th differential coefficient of G_m at $w = 0$ in (40) is described by a series

$$(G_m^{(n-k)})_{w=0} = \sum_{l=0}^{n-k}\frac{(n-k)!}{(n-k-l)!\,l!}(G_{m-1}^{(n-k-l)})_{w=0}(h^{(l)})_{w=0}$$

$$= \sum_{l=0}^{n-k}\frac{(n-k)!}{(n-k-l)!}p(l, t)(G_{m-1}^{(n-k-l)})_{w=0} \tag{42}$$

because

$$p(l,t)=\frac{1}{l!}\left(\frac{\partial^l h}{\partial w^l}\right)_{w=0} \tag{43}$$

We introduce the functions

$$K_m^{(k)}=\frac{1}{k!}(G_m^{(k)})_{w=0} \tag{44}$$

and

$$Q_0^{(k)}=\frac{1}{k!}(Q^{(k)})_{w=0}. \tag{45}$$

Equation (40) is expressed as

$$P(m;n,t)=\sum_{k=0}^{n} K_m^{(n-k)}Q_0^{(k)} \tag{46}$$

where

$$K_m^{(n-k)}=\sum_{l=0}^{n-k} p(l,t)K_{m-1}^{(n-k-l)} \tag{47}$$

which is obtained successively using the relation

$$K_0^{(k)}=\delta_{k,0}. \tag{48}$$

The term $Q_0^{(k)}$ will be calculated when a more definite model is given.

4.2. Counted particles

We next consider the conditional probability $P_m(k;n,t)$ that m counts have been recorded during a time interval $(0,t)$ and n particles are found in the medium at time $t>0$ after we had k particles at $t=0$. Repeating the familiar procedure, we obtain the expression for the PGF of $P_m(k;n,t)$

$$T(u,v,w,t)=\sum_{k=0}^{\infty} u^k \sum_{m=0}^{\infty} v^m \sum_{n=0}^{\infty} w^n P_m(k;n,t) \tag{49}$$

as

$$T(u,v,w,t)=\frac{1}{1-ug(v,w,t)}R(v,w,t) \tag{50}$$

where

$$R(v,w,t)=\exp\left(\int_0^t S(\tau)(g(v,w,t-\tau)-1)\,\mathrm{d}\tau\right). \tag{51}$$

The coefficient $R(v,w,t)$ in (50) is the effect of source particles appearing in $(0,t)$ and the other part of $T(u,v,w,t)$ is that of particles existing at $t=0$.

The conditional probability $P_m(k;n,t)$ is obtained similarly as before;

$$P_m(k;n,t)=\sum_{i=0}^{m}\sum_{j=0}^{n} K_k^{(m-i,n-j)}R_0^{(i,j)} \tag{52}$$

where

$$R_0^{(i,j)} = \frac{1}{i!j!}(R^{(i,j)})_{v=w=0} \tag{53}$$

and

$$K_k^{(m-i,n-j)} = \sum_{l_1=0}^{m-i} \sum_{l_2=0}^{n-j} p_{l_1}(l_2, t) K_{k-1}^{(m-i-l_1, n-j-l_2)}. \tag{54}$$

The term $R^{(i,j)}$ in equation (53) represents the differential coefficient of $R(v, w, t)$ of ith order on v and jth order on w. The coefficient $K_k^{(m-i,n-j)}$ for $k > 0$ can be calculated successively using the relation

$$K_0^{(i,j)} = \delta_{i,0}\delta_{j,0}. \tag{55}$$

5. Application

In the above formulation, the branching is described by a distribution function p_ν. Usually, p_ν is unknown. In the present section, we suppose that exactly two particles are produced by a multiplicative reaction, which is a particular but very important and useful branching mode, and which has been studied by several workers (Feller 1939, Kendall 1948). The distribution function p_ν is, therefore,

$$p_\nu = \begin{cases} 1 & (\nu = 2) \\ 0 & (\nu \neq 2) \end{cases} \tag{56}$$

and its PGF is given by

$$\psi(w) = w^2. \tag{57}$$

5.1. Branching started from a single particle

5.1.1. Existing particles. Equation (6) is written as

$$\begin{aligned} \Phi(w) &= \lambda_m w^2 - \lambda_t w + \lambda_c \\ &= \lambda_m (w-1)(w-\beta) \end{aligned} \tag{58}$$

where

$$\beta = \lambda_c / \lambda_m. \tag{59}$$

(i) Case of $\beta \neq 1$ (non-critical case). The function $k(w)$ defined by (10) is, in this case,

$$k(w) = \frac{1}{\alpha} \ln \left| \frac{w-\beta}{w-1} \right| \tag{60}$$

where

$$\alpha = \lambda_c - \lambda_m. \tag{61}$$

The inverse function of $k(w)$ is found from (60) as

$$F(w) = \frac{\exp(\alpha w) \pm \beta}{\exp(\alpha w) \pm 1}. \tag{62}$$

Consequently the PGF shown in (9) is

$$h(w,t)=\frac{(1-\beta\exp(-\alpha t))w-\beta(1-\exp(-\alpha t))}{(1-\exp(-\alpha t))w-(\beta-\exp(-\alpha t))} \tag{63}$$

which satisfies (7). The probability $p(n,t)$ is given, using (43), by

$$p(n,t)=\begin{cases}\dfrac{\beta(1-\exp(-\alpha t))}{\beta-\exp(-\alpha t)} & (n=0)\\ (\beta-1)^2\,\mathrm{e}^{-\alpha t}\dfrac{(1-\exp(-\alpha t))^{n-1}}{(\beta-\exp(-\alpha t))^{n+1}} & (n\geqslant 1).\end{cases} \tag{64}$$

The mean number of particles found in the medium is

$$E(n)=\left(\frac{\partial h}{\partial w}\right)_{w=1}=\mathrm{e}^{-\alpha t} \tag{65}$$

and the variance of n is given by

$$\begin{aligned}\sigma_1^2&=E[n(n-1)]+E(n)-E(n^2)\\ &=\frac{\beta+1}{\beta-1}\mathrm{e}^{-\alpha t}(1-\mathrm{e}^{-\alpha t}).\end{aligned} \tag{66}$$

(ii) Case of $\beta=1$ (critical case). Because

$$\Phi(z)=\lambda_\mathrm{m}(w-1)^2. \tag{67}$$

in this case,

$$k(w)=-\frac{1}{\lambda_\mathrm{m}}\frac{1}{w-1}. \tag{68}$$

In a similar way to the previous case,

$$h(w,t)=\frac{(\lambda_\mathrm{m}t-1)w-\lambda_\mathrm{m}t}{\lambda_\mathrm{m}tw-(\lambda_\mathrm{m}t+1)} \tag{69}$$

and

$$p(n,t)=\begin{cases}\left(1+\dfrac{1}{\lambda_\mathrm{m}t}\right)^{-1} & (n=0)\\ \dfrac{1}{(\lambda_\mathrm{m}t)^2}\left(1+\dfrac{1}{\lambda_\mathrm{m}t}\right)^{-n-1} & (n\geqslant 1).\end{cases} \tag{70}$$

The mean number of particles and the variance are, respectively,

$$E(n)=1 \tag{71}$$

and

$$\sigma_1^2=2\lambda_\mathrm{m}t. \tag{72}$$

The variance increases with time.

5.1.2. Counted particles. Using the function $\phi(u,v)$ for this case

$$\phi(v,w)=\lambda_\mathrm{m}w^2-\lambda_t w+\lambda_\mathrm{a}+\lambda_\mathrm{d}v \tag{73}$$

the PGF shown in (19) is derived by the familiar procedure

$$g(v, w, t) = \frac{(\zeta - \eta \exp(-\theta t))w - \eta\zeta(1 - \exp(-\theta t))}{(1 - \exp(-\theta t))w - (\eta - \zeta \exp(-\theta t))} \tag{74}$$

where

$$\theta = [(\lambda_c - \lambda_m)^2 + 4\lambda_d\lambda_m(1 - v)]^{1/2} \tag{75}$$

$$\eta = \frac{\lambda_t + \theta}{2\lambda_m} \qquad \zeta = \frac{\lambda_t - \theta}{2\lambda_m} \tag{76}$$

and

$$\lambda_c = \lambda_a + \lambda_d. \tag{77}$$

The probability is estimated from the following equation:

$$p_m(n, t) = \begin{cases} \dfrac{1}{m!}\left[\dfrac{\partial^m}{\partial v^m}\left(\dfrac{\eta\zeta[1 - \exp(-\theta t)]}{\eta - \zeta \exp(-\theta t)}\right)\right]_{v=0} & (n = 0) \\ \dfrac{1}{m!}\left[\dfrac{\partial^m}{\partial v^m}\left((\eta - \zeta)^2 \, e^{-\theta t} \dfrac{[1 - \exp(-\theta t)]^{n-1}}{[\eta - \zeta \exp(-\theta t)]^{n+1}}\right)\right]_{v=0} & (n \geq 1). \end{cases} \tag{78}$$

We cannot find the general form of the mth differential coefficient at $v = 0$, but there will be no difficulty for numerical estimation of the probability, because the detection efficiency, i.e. λ_d, is usually small and estimation of the probability is, therefore, necessary only for small m.

The PGF of $p_m(t)$ shown in (21) is described, using (74), by

$$f(v, t) = \frac{\eta(1 - \zeta) \exp(-\theta t) + \zeta(\eta - 1)}{(1 - \zeta) \exp(-\theta t) + (\eta - 1)}. \tag{79}$$

5.2. Branching with random immigration

5.2.1. Existing particles. The source strength $S(t)$ is assumed to be a constant S.

(i) Case of $\beta \neq 1$ (non-critical case). From simple calculations using (23) and (63), the PGF is obtained as

$$H(w, t) = \exp\left(S\frac{\beta - 1}{\alpha} \ln \frac{\beta - 1}{(\beta - \exp(-\alpha(t - t_0))) - (1 - \exp(-\alpha(t - t_0)))w}\right). \tag{80}$$

The probability $P(n, t)$ is calculated easily from the PGF so that

$$P(n, t) = \begin{cases} \exp\left(\dfrac{S}{\alpha}(\beta - 1) \ln \dfrac{\beta - 1}{\beta - \exp(-\alpha(t - t_0))}\right) & (n = 0) \\ \dfrac{1}{n!}\dfrac{S}{\alpha}(\beta - 1)\left(\dfrac{S}{\alpha}(\beta - 1) + 1\right) \cdots \left(\dfrac{S}{\alpha}(\beta - 1) + n - 1\right) & \\ \quad \times \exp\left(\dfrac{S}{\alpha}(\beta - 1) \ln \dfrac{\beta - 1}{\beta - \exp(-\alpha(t - t_0))}\right) & (n \geq 1). \end{cases} \tag{81}$$

The mean number of particles found in the medium and the variance are, respectively,

$$E(n) = \frac{S}{\alpha}(1 - \exp(-\alpha(t - t_0))) \tag{82}$$

and

$$\sigma_2^2=\frac{S}{\alpha}\frac{1}{\beta-1}[1-\exp(-\alpha(t-t_0))][\beta-\exp(-\alpha(t-t_0))]. \tag{83}$$

(ii) Case of $\beta=1$ (critical case). Using (23) and (69),

$$H(w,t)=\exp\left(-\frac{S}{\lambda_m}\ln[\lambda_m(t-t_0)(1-w)+1]\right) \tag{84}$$

$$P(n,t)=\begin{cases}\exp\left(-\frac{S}{\lambda_m}\ln(\lambda_m(t-t_0)+1)\right) & (n=0)\\ \frac{1}{n!}\frac{S}{\lambda_m}\left(\frac{S}{\lambda_m}+1\right)\dots\left(\frac{S}{\lambda_m}+n-1\right)\left(\frac{\lambda_m(t-t_0)}{\lambda_m(t-t_0)+1}\right)^n & \\ \quad\times\exp\left(-\frac{S}{\lambda_m}\ln(\lambda_m(t-t_0)+1)\right) & (n\geqslant 1)\end{cases} \tag{85}$$

$$E(n)=S(t-t_0) \tag{86}$$

and

$$\sigma_2^2=S(t-t_0)(\lambda_m(t-t_0)+1). \tag{87}$$

5.2.2. Counted particles. (i) Case of $\beta\neq 1$ (non-critical case). Using (79) and (80), the PGF shown in (31) is given by

$$L(v,t,\Delta t)=\exp\left(\frac{\beta-1}{\alpha}S\ln A(v,t,\Delta t)\right)\exp(SI_1) \tag{88}$$

where

$$A(v,t,\Delta t)=\frac{(\beta-1)[(1-\zeta)\,e^{-\theta\Delta t}+\eta-1]}{(1-\zeta)[\beta-\eta+(\eta-1)\,e^{-\alpha(t-t_0)}]\,e^{-\theta\Delta t}+(\eta-1)[\beta-\zeta-(1-\zeta)\,e^{-\alpha(t-t_0)}]} \tag{89}$$

and

$$I_1=\int_t^{t+\Delta t}[f(v,t+\Delta t-\tau)-1]\,d\tau$$
$$=(1-\zeta)(e^{\theta\Delta t}-1)+\frac{\eta+\zeta-2}{\theta}\ln\left(\frac{\eta-\zeta}{(1-\zeta)\exp(-\theta\Delta t)+\eta-1}\right). \tag{90}$$

(ii) Case of $\beta=1$ (critical case). Using (79) and (84), the PGF is shown as

$$L(v,t,\Delta t)=\exp\left(-\frac{S}{\lambda_m}\ln A(v,t,\Delta t)\right)\exp(SI_1) \tag{91}$$

where

$$A(v,t,\Delta t)=1+\lambda_m(t-t_0)\left(\frac{(\eta-1)(1-\zeta)(1-e^{-\theta\Delta t})}{(1-\zeta)\,e^{-\theta\Delta t}+\eta-1}\right) \tag{92}$$

and

$$I_1=\frac{1}{2\lambda_m}\theta(e^{\theta\Delta T}-1). \tag{93}$$

5.3. *Time series representations*

5.3.1. Existing particles. (i) Case of $\beta \neq 1$ (non-critical case). From (38) and (63), the PGF is obtained as

$$G(u, w, t)=\frac{(1-w)\exp(-\alpha t)-(\beta-w)}{(1-\beta u)(1-w)\exp(-\alpha t)-(1-u)(\beta-w)} \times \exp\left[S\frac{(\beta-1)}{\alpha}\ln\left(\frac{\beta-1}{(\beta-w)-(1-w)\exp(-\alpha t)}\right)\right]. \tag{94}$$

From simple calculations, the differential coefficients shown in (45) are

$$Q_0^{(k)}=\begin{cases}\exp\left(S\frac{(\beta-1)}{\alpha}\ln\frac{\beta-1}{\beta-\exp(-\alpha t)}\right) & (k=0)\\ \frac{1}{k!}S\frac{\beta-1}{\alpha}\left(S\frac{\beta-1}{\alpha}+1\right)\ldots\left(S\frac{\beta-1}{\alpha}+k-1\right)\left(\frac{1-\exp(-\alpha t)}{\beta-\exp(-\alpha t)}\right)^k & \\ \quad\times\exp\left(S\frac{\beta-1}{\alpha}\ln\frac{\beta-1}{\beta-\exp(-\alpha t)}\right) & (k\geqslant 1).\end{cases} \tag{95}$$

The conditional probability $P(m; n, t)$ can be estimated by using (46), (47) and (95).

5.3.2. Case of $\beta = 1$ (critical case). In a similar way

$$G(u, w, t)=\frac{1+\lambda_m t(1-w)}{\lambda_m t(1-u)(1-w)+1-uw}\exp\left(-\frac{S}{\lambda_m}\ln[1+\lambda_m t(1-w)]\right) \tag{96}$$

and

$$Q_0^{(k)}=\begin{cases}\exp\left(-\frac{S}{\lambda_m}\ln(1+\lambda_m t)\right) & (k=0)\\ \frac{1}{k!}\frac{S}{\lambda_m}\left(\frac{S}{\lambda_m}+1\right)\ldots\left(\frac{S}{\lambda_m}+k-1\right)\left(1+\frac{1}{\lambda_m t}\right)^{-k} & \\ \quad\times\exp\left(-\frac{S}{\lambda_m}\ln(1+\lambda_m t)\right) & (k\geqslant 1).\end{cases} \tag{97}$$

5.3.3. Counted particles. Using (50), (51) and (74), the PGF and the coefficient $R(v, w, t)$ are easily obtained as

$$T(u, v, w, t)=\frac{(1-e^{-\theta t})w-(\eta-\zeta e^{-\theta t})}{[(1-\zeta u)-(1-\eta u)e^{-\theta t}]w-[\eta(1-\zeta u)-\zeta(1-\eta u)e^{-\theta t}]} \times\exp\left[(\zeta-1)St+\frac{S}{\lambda_m}\ln\left(\frac{\eta-\zeta}{(e^{-\theta t}-1)w+\eta-\zeta e^{-\theta t}}\right)\right] \tag{98}$$

and

$$R(v, w, t)=\exp\left[(\zeta-1)St+\frac{S}{\lambda_m}\ln\left(\frac{\eta-\zeta}{(e^{-\theta t}-1)w+\eta-\zeta e^{-\theta t}}\right)\right]. \tag{99}$$

Introducing the function $B_j(v, t)$ given by

$$B_j(v,t)=\begin{cases}\exp\left[(\zeta-1)St+\dfrac{S}{\lambda_m}\ln\left(\dfrac{\eta-\zeta}{\eta-\zeta\,e^{-\theta t}}\right)\right] & (j=0)\\ \dfrac{1}{j!}\dfrac{S}{\lambda_m}\left(\dfrac{S}{\lambda_m}+1\right)\dots\left(\dfrac{S}{\lambda_m}+j-1\right)\left(\dfrac{1-e^{-\theta t}}{\eta-\zeta\,e^{-\theta t}}\right)^j & \\ \qquad\times\exp\left[(\zeta-1)St+\dfrac{S}{\lambda_m}\ln\left(\dfrac{\eta-\zeta}{\eta-\zeta\,e^{-\theta t}}\right)\right] & (j\geqslant 1)\end{cases} \tag{100}$$

the coefficient $R_0^{(i,j)}$ represented by (53) is described as

$$R_0^{(i,j)}=\frac{1}{i!}\left(\frac{\partial^i}{\partial v^i}B_j(v,t)\right)_{v=0}. \tag{101}$$

The order i in (101) is the number of counts due to source particles appearing in $(0, t)$. It is, therefore, sufficient to estimate $R_0^{(i,j)}$ only for small i when the detection efficiency is small.

5.4. *Non-binary branching*

We have considered, in the above, the statistics of binary branching processes. In the case of general branching, the distribution function p_ν is usually unknown as described before. It is, however, possible to estimate the moments of found or counted particle number when the mean number and variance of particles produced by a multiplicative reaction are known.

Expanding $\psi(w)$ around $w=1$,

$$\psi(w)\doteqdot 1+\bar{\nu}(w-1)+\frac{\overline{\nu(\nu-1)}}{2}(w-1)^2 \tag{102}$$

where $\bar{\nu}$ and $\overline{\nu(\nu-1)}$ are the means of ν and $\nu(\nu-1)$, respectively. Equations (6) and (15) are then expressed as

$$\Phi(w)=\frac{\overline{\nu(\nu-1)}}{2}\lambda_m w^2-[\lambda_t+(\overline{\nu(\nu-1)}-\bar{\nu})\lambda_m]w+\lambda_c+\left(\frac{\overline{\nu(\nu-1)}}{2}-\bar{\nu}+1\right)\lambda_m \tag{103}$$

and

$$\phi(v,w)=\frac{\overline{\nu(\nu-1)}}{2}\lambda_m w^2-[\lambda_t+(\overline{\nu(\nu-1)}-\bar{\nu})\lambda_m]w+\lambda_a+\left(\frac{\overline{\nu(\nu-1)}}{2}-\bar{\nu}+1\right)\lambda_m+\lambda_d v \tag{104}$$

respectively. Comparing (103) with (58) and (104) with (73), we notice that all of the PGF obtained for binary branching are applicable to a non-binary case if the reaction rates are replaced as follows:

$$\lambda_m\leftarrow\frac{\overline{\nu(\nu-1)}}{2}\lambda_m \tag{105}$$

and

$$\lambda_c \leftarrow \lambda_c + \left(\frac{\overline{\nu(\nu-1)}}{2} - \bar{\nu} + 1 \right) \lambda_m \qquad \text{(for existing particle statistics)} \tag{106}$$

$$\lambda_a \leftarrow \lambda_a + \left(\frac{\overline{\nu(\nu-1)}}{2} - \bar{\nu} + 1 \right) \lambda_m \qquad \text{(for counted particle statistics).} \tag{106$'$}$$

These PGF are correct only around $w = 1$, and, therefore, can be used only for estimating the moments of particle number. The probabilities given by (64), etc, cannot be estimated with them.

When the detection efficiency is small, the PGF given by (88) and (91) are, however, applicable for estimating the probabilities as well as for estimating the moments. In this case, the probability of counting one or more particles is exceedingly small and, therefore, the function $f(v, \Delta t)$ is very close to unity. Consequently, in order to estimate $H[f(v, \Delta t), t]$ in (31), it is sufficient to know $h(w, t)$ only around $w = 1$. The other coefficient in (31) can be estimated correctly from the function $f(v, t)$ which is obtained from $g(v, w, t)$ by setting $w = 1$ as shown in (21). It is, therefore, considered that (88) and (91) are applicable for estimating both the moments and probabilities in non-binary branching procesees if the reaction rates are replaced as in (105) and (106′).

References

Bell G I 1965 *Nucl. Sci. Eng.* **21** 390
Bellman R and Harris T E 1952 *Ann. Math.* **55** 280
Carruthers P and Shin C C 1985 *Phys. Lett.* **165B** 209
Feller W 1939 *Acta Biotheor.* **5** 11
Furukawa H 1986 *Phys. Rev.* A **34** 2315
Giovannini A 1979 *Nucl. Phys.* B **161** 429
Hering H 1973 *Adv. Appl. Prob.* **5** 391
Heyde C C 1986 *J. Appl. Prob.* A **23** 345
Kendall D G 1948 *Ann. Math. Stat.* **19** 1
Kobayashi T 1968 *J. Nucl. Sci. Tech.* **5** 145
Lin D L 1979 *Phys. Rev.* B **20** 5238
—— 1981 *J. Appl. Phys.* **52** 3575
Lushinikov A A, Tokar Ya I and Tsitskishivili M S 1981 *Dokl. Phys. Chem.* (*USSR*) **256** 1155
Pakes A G 1986 *Adv. Appl. Prob.* **18** 628
Sevastyanov B A 1957 *Theor. Prob. Appl.* **2** 339
Shin C C 1986 *Phys. Rev.* D **33** 339